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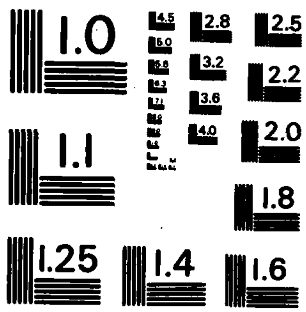

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STRUCTURES REPORT 395

MAXIMUM LIKELIHOOD ESTIMATES OF LINEAR DYNAMIC SYSTEM PARAMETERS

by

T. G. RYALL

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T. G. RYALL

SUMMARY

The parameters of a linear dynamic system are estimated by using the maximum likelihood method.

Maximum likelihood estimates are asymptotically unbiased and efficient, that is they are "good" estimates. Furthermore the maximum likelihood estimate of a function of parameters is the function of the maximum likelihood estimates of those parameters. Two different situations are studied:

(1) the excitation force is random;
(2) the excitation force is deterministic.

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NOTATION

Drawing a distinction between a random variable and a possible realisation of that random variable leads to a cumbersome notation.

Contextual information should make it clear which possibility is referred to. As a result, in this report no distinction will be made.

$\underline{M}$	A matrix with components M_{ij}
$\mathbf{x}$	A vector with components x_i
E	The mathematical expectation operator
$\sum_{\omega}$	A summation over all observed frequencies
$\prod_{\omega}$	A product over all observed frequencies
$\mathbf{u}^*$	The complex conjugate of $\mathbf{u}$
$\mathbf{u}^T$	The transpose of $\mathbf{u}$
n	The number of ensemble averages
m	The number of discrete frequencies observed

FORMAL DEFINITIONS

$O^{(j)}{}_{(i\omega)} = j$th observation of output at frequency ω.

$I^{(j)}{}_{(i\omega)} = j$th observation of input at frequency ω.

Sample estimate of average output power

$$P_o(i\omega) = \frac{1}{N}\sum_{j=1}^{N} O^{(j)}{}_{(i\omega)} O^{(j)*}{}_{(i\omega)}$$

Sample estimate of average input power

$$P_I(i\omega) = \frac{1}{N}\sum_{j=1}^{N} I^{(j)}{}_{(i\omega)} I^{(j)*}{}_{(i\omega)}$$

Sample estimate of average cross power

$$P_c(i\omega) = \frac{1}{N}\sum_{j=1}^{N} O^{(j)}{}_{(i\omega)} I^{(j)*}{}_{(i\omega)}$$

Sample estimate of coherence

$$\gamma^2_{(i\omega)} = \frac{P_c \cdot P_c^*}{P_I \cdot P_o}$$

1. INTRODUCTION

This report is concerned with the maximum likelihood estimation of dynamic system parameters. The particular linear systems considered are:

(i) an elastic aircraft on the ground;

(ii) an elastic aircraft in flight.

(The method used, however, is of quite general application.) The aerodynamic feedback forces depend in a linear way on the past history of the aircraft. The transfer functions for the "aircraft-air" system cannot be represented by rational transfer functions. However, rational transfer functions can approximate non-rational transfer functions to any degree of accuracy over a limited frequency range. Chapter 2 gives a brief summary of the properties of linear systems pertinent to the estimation process. Chapter 3 formulates the deterministic likelihood function for excitation for single and multiple channels.

"Globally" stable numerical algorithms are also presented to solve the likelihood equations and an estimate is made of the asymptotic covariance matrix.

Chapter 4 formulates the likelihood function for random excitation of single and multiple channels. A "globally" stable numerical algorithm is presented for the single channel case.

No attempt is made to solve the general inverse problem of determining the mass density variation throughout the structure nor the boundary conditions. There is also no attempt made at finding coefficients of a set of linear differential equations which is a minimal realisation of the process.

The purpose of the estimation procedure is to provide estimates of the damped modes and frequencies which may be compared with the results of finite element program and/or allow the calculation of generalised mass, generalised damping and generalised stiffness. The Ho algorithm (Anderson and Moore) provides a method to obtain a minimal realisation from a transfer function matrix. It is well known that the general inverse problem is unstable, i.e. large changes in mass density variations can lead to small changes in the impulse response. An extremely good summary of the maximum likelihood method is given by Kendall and Stuart [1]. The asymptotic distribution of the finite Fourier transform, on which this method of estimation relies for robustness is covered by Brillinger [2] and Anderson [3]. Only elementary result in dynamics theory are called for with the exception of the minimum-phase transfer functions and the fact that the principal subdeterminants of the transfer function matrix under certain conditions form a minimum-phase system.

There can be a number of reasons for estimating dynamic system parameters. The simple answer being to obtain a mathematical model, however this really begs the question. Given a mathematical model of an aircraft on the ground it is possible to compute a model for an aircraft in flight. Stability questions may then be answered, i.e. for what range of values of forward speed and altitude is the aircraft stable? Control laws may also be devised to stabilize an unstable system or to move the system in the shortest possible time (subject to control constraints) to another state. The last problem requires an accurate model of the dynamical system. The literature on system identification is immense; a good survey paper is *System Identification—A Survey*, by K. J. Astrom and P. Dykhoff. The paper *Spectrum Analysis—A Modern Perspective* by S. M. Kay and S. L. Marple Jr. looks at some of the deficiencies of various estimation procedures, especially in the time domain, and suggests that the direction of future research is in formulating computationally efficient maximum likelihood auto-regression moving averages for spectral estimation. This report is confined to the frequency domain and as such has a number of inherent advantages for linear systems:

(i) lightly damped modes "separate" more than in the time domain;

(ii) measurement noise and process noise are much easier to model;

(iii) the Fourier transform being a sum of a large number of sample time values allows the application of the central limit theorem to give a Gaussian distribution for both the real and the imaginary components—in other words there is a statistical robustness;

(iv) if the noise processes are stationary each frequency component is asymptotically independently distributed.

The disadvantages of frequency domain analysis are:

(i) non-linearities are harder to deal with in the frequency domain;

(ii) non-stationarity of noise causes frequency components to be statistically correlated.

2. LINEAR SYSTEMS

In this section a brief summary will be given of the essential properties of linear systems pertinent to the estimation process.

Consider a second-order linear stable system

$$\underline{\underline{M}}\ddot{\mathbf{x}}+\underline{\underline{D}}\dot{\mathbf{x}}+\underline{\underline{K}}\mathbf{x} = 0 \tag{2.1}$$

where $\mathbf{x}$ may be finite dimensional or infinite dimensional.

If an impulsive force is applied at the ith position it follows that

$$\underline{\underline{M}}\ddot{\mathbf{x}}+\underline{\underline{D}}\dot{\mathbf{x}}+\underline{\underline{K}}\mathbf{x} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \delta(t) \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \; i\text{th position} \tag{2.2}$$

Equation (2.2) is equivalent to (2.1) for $t > 0+$ with initial conditions $x(0+) = 0$

$$x(0+) = 0 \qquad M\dot{x}(0+) = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \; i\text{th position.}$$

The response at time t is

$$\mathbf{x}(t) = \sum_i (k_i\mathbf{u}_i e^{\alpha_i t}+k_i^*\mathbf{u}_i^* e^{\alpha_i^* t}) \tag{2.3}$$

where $\mathbf{u}$'s and α's satisfy

$$[\alpha^2\underline{\underline{M}}+\alpha\underline{\underline{D}}+\underline{\underline{K}}]\mathbf{u} = 0. \tag{2.4}$$

If impulses are applied at all other positions then the impulse response matrix is, assuming that the matrix is symmetric,

$$\underline{\underline{X}}(t) = \sum_{i=1}^{\infty} (\mathbf{u}_i\mathbf{u}_i^{\mathrm{T}} e^{\alpha_i t}+\mathbf{u}_i^*\mathbf{u}_i^{*\mathrm{T}} e^{\alpha_i^* t}). \tag{2.5}$$

If the matrix is not symmetric then

$$\underline{\underline{X}}(t) = \sum_{i=1}^{\infty} (\mathbf{u}_i\mathbf{v}_i^{\mathrm{T}} e^{\alpha_i t}+\mathbf{u}_i^*\mathbf{v}_i^{*\mathrm{T}*} e^{\alpha_i^* t}) \tag{2.6}$$

where $\mathbf{v}$ satisfies

$$[\alpha^2\underline{\underline{M}}+\alpha\underline{\underline{D}}+\underline{\underline{K}}]^{\mathrm{T}}\mathbf{v} = 0.$$

For a linear system with general input $\mathbf{I}(t)$

$$\mathbf{O}(t) = \int_0^t \underline{\underline{X}}(t-\tau)\mathbf{I}(\tau)dt \tag{2.7}$$

where $\mathbf{O}(t)$ is the reponse vector. On taking the Fourier transform of equation (2.7) it follows that

$$O_{(i\omega)} = \underline{\underline{T}}_{(i\omega)} I_{(i\omega)} \tag{2.8}$$

where $T_{(i\omega)}$ is the transfer function matrix. $T_{(i\omega)}$ is also known as the Fourier transform of the Green's function. Equations (2.5), (2.6), (2.7) and (2.8) are very general results and do not rely of the "lumped parameter" model (2.1) which assumes a spatial averaging of the true partial differential equation and that the system has no memory. The probability density functions of observed time functions become much simpler in the frequency domain than in the time domain hence all calculations are done in the frequency domain. The parameters to be estimated are the damped modes (modes of decay), decay rates and decay frequencies. A number of situations are studied:

(i) linear system excited by a deterministic signal;

(ii) linear system excited by a random signal.

3. DETERMINISTIC EXCITATION

3.1 Deterministic Excitation, Single Input and Single Output

In this Section a linear dynamic system is excited by a deterministic force at a particular point and the response is measured at the same or another point. Typical force inputs that may be used are:

(i) $F(t) = A \sin \dfrac{\omega}{2T} t^2 \qquad 0 \leqslant t \leqslant T;$

(ii) $F(t) =$ "Hammer blow".

Any force history that has an approximately flat power spectrum over the frequency range of interest may be used. The relationship between measured output and measured input is

$$O_{(i\omega)} = T_{(i\omega)} I_{(i\omega)} + N_{(i\omega)}, \tag{3.1.1}$$

where $N_{(i\omega)}$ is the noise at frequency ω and $E(N_{(i\omega)}) = 0$.

$$E(N_{(i\omega)} N^*_{(i\omega)}) = \sigma^2(\omega). \tag{3.1.2}$$

The noise may be considered to be made up of three components, viz:

(i) output measurement noise;

(ii) effect of non-linearities (non-linearities are treated as though they come from a random noise source),

(iii) process noise.

In the time domain nothing can be said about the probability density function of the noise, without making some very strong assumptions. In the frequency domain, however, it is assymptotically true that the number of points in the transform increases:

(i) that the real and imaginary parts of $N_{(i\omega)}$ have a normal probability density function;

(ii) each frequency component is independently distributed.

Statement (i) has two exceptions. At zero frequency and at the NYQIST frequency the imaginary part of $N_{(i\omega)}$ is zero with probability one. These two frequencies will be excluded from this analysis although only simple modifications are needed to include them. The likelihood function may be formed in the following manner, at each frequency ω the output has the "complex" normal probability density function

$$\frac{1}{2\Pi\sigma^2(\omega)}(\exp -\{(O-TI)(O^*-T^*I^*)/\sigma^2(\omega)\})$$

where $\sigma^2(\omega)$ is the unknown noise power at ω. The likelihood function, when there are n ensembles, is at a particular frequency ω

$$\left(\frac{1}{2\pi}\right)^n\left[\frac{1}{\sigma^2(\omega)}\right]^n(\exp-\{\Sigma(O-TI)(O^*-T^*I^*)/\sigma^2(\omega)\})$$

The total likelihood function for n ensembles across a discrete frequency spectrum is

$$L=\prod_{\omega}\left(\frac{1}{2\pi}\right)^n\left[\frac{1}{\sigma^2(\omega)}\right]^n\exp-\{\Sigma(O-TI)(O^*-T^*I^*)/\sigma^2(\omega)\}. \tag{3.1.3}$$

The likelihood function is a maximum when the negative log likelihood is a minimum. Not only does L contain the parameters of the transfer function, L also contains the noise spectrum $\sigma^2(\omega)$.

Now

$$\frac{\partial L}{\partial\sigma^2(\omega)}=0$$

when

$$\sigma^2(\omega)=\frac{1}{n}\Sigma(O-TI)(O-TI)^*. \tag{3.1.4}$$

Substitution of (3.1.4) into (3.1.3) and taking the negative logarithm leads to the minimisation of

$$\sum_{\omega}\log\left\{\Sigma\frac{1}{n}(O-TI)(O^*-T^*I^*)\right\}. \tag{3.1.5}$$

A simple interpretation may be put on the function (3.1.5), viz. choose parameters in the transfer function such as to minimise the estimate of the total noise in decibels.

3.2 Numerical Solution of the Objective Function

In any optimisation procedure the objective function should be well scaled. Let the parameters (decay rates, decay frequencies and damped modes) of interest be $\alpha_1,\ldots,\alpha_p$. Then the function to be minimised is

$$F(\alpha_1,\ldots,\alpha_p)=\frac{1}{m}\sum_{\omega}\log\left[\frac{1}{N}\Sigma(O-TI).(O-TI)^*\right]$$

where m is the number of frequency points. The non-linear equations to be solved are

$$\frac{\partial F}{\delta\alpha_l}=0\qquad l=1,\ldots,p \tag{3.2.1}$$

The Newton-Raphson method of solving non-linear equations states that if α_i are initial estimates then $\alpha_i+\Delta\alpha_i$ are improved estimates where

$$\frac{\partial F}{\partial\alpha_i}+\Sigma\frac{\partial^2F}{\partial\alpha_i\partial\alpha_j}\Delta\alpha_j=0\qquad\text{for all } i \tag{3.2.2}$$

Unfortunately this method fails to converge or takes a long time to converge unless the initial estimates are sufficiently close to the true values so as to ensure that the Hessian $\dfrac{\partial^2F}{\partial\alpha_i\partial\alpha_j}$ matrix is positive definite. An approximation to the Hessian matrix is used which is globally positive semi-definite. This approximation is the asymptotic expected value of the Hessian matrix

$$\frac{\partial F}{\partial\alpha_i}=\frac{1}{M}\sum_{\omega}\frac{-\dfrac{\partial T}{\partial\alpha_l}(P_*^*-T^*P_1^*)-\dfrac{\partial T^*}{\partial\alpha_l}(P_*-TP_1)}{\dfrac{1}{N}\Sigma(O-TI)(O-TI)^*}. \tag{3.2.3}$$

$$\frac{\partial^2 F}{\partial\alpha_i\partial\alpha_j} \doteq \frac{1}{M}\sum_{\omega}\frac{P_i\left\{\frac{\partial T}{\partial\alpha_i}\frac{\partial T^*}{\partial\alpha_j}+\frac{\partial T}{\partial\alpha_j}\frac{\partial T^*}{\partial\alpha_i}\right\}}{\frac{1}{N}\Sigma(O-TI)(O-TI)^*} \tag{3.2.4}$$

The asymptotic covariance matrix which reflects the amount of uncertainty in the estimates is given by

$$\frac{1}{MN}\left[\frac{\partial^2 F}{\partial\alpha_i\partial\alpha_j}\right]^{-1}.$$

3.3 Deterministic Single Point Excitation Multiple Point Measurement

If a linear system is excited at a single point and measurements are mae at a number of points then there are two possibilities which will be discussed:

(i) the outputs are measured simultaneously;

(ii) the outputs are not measured simultaneously.

If the outputs are measured simultaneously then it is possible that the system output is contaminated by correlated noise. Under these circumstances the maximum likelihood method leads to the minimisation of the "integral" of the logarithm of the determinant of the estimated covariance matrix of the noise. The function to be minimised in the case of two outputs is

$$\sum_{\omega}\log\begin{vmatrix}\overline{(O_1-T_{11}I_1)(O_1^*-T_{11}^*I_1^*)} & \overline{(O_1-T_{11}I_1)(O_2^*-T_{21}^*I_1^*)}\\ \overline{(O_2-T_{21}I_1)(O_1^*-T_{11}^*I_1)} & \overline{(O_2-T_{21}I_1)(O_2^*-T_{21}^*I_1^*)}\end{vmatrix}. \tag{3.3.1}$$

Sufficient statistics for this situation are: all the average output powers and all the average cross-powers between outputs and all the average cross-powers between output and input. If the outputs are not measured simultaneously and are not measured during the same excitation then the correlated noise is assumed not to be present and the off-diagonal terms in (3.3.1) vanish.

The maximum likelihood method under these circumstances leads to the minimisation of the sum of all the "integrated" logarithms of the estimated noise power spectra. This result is the direct analogue of the single input–single output case.

4. RANDOM EXCITATION

4.1 Single Output

If a system is excited by random noise then both real and imaginary components of the output have a normal distribution. Each frequency component is asymptotically independently distributed. If the real and imaginary components of the input are independent and have the same expected power then phase has a circular uniform distribution and output power has a negative exponential probability density function. The previous restriction is equivalent to assuming that the process has a certain type of stationarity.

Since

$$\underline{O} = T\underline{I} \tag{4.1.1}$$

it follows that

$$OO^* = TT^*II^* \tag{4.1.2}$$

Hence the likelihood function at a particular frequency is

$$L = \frac{1}{P}\exp\left[-\frac{OO^*}{II^*}\frac{1}{P}\right] \tag{4.1.3}$$

where

$$P = TT^*. \tag{4.1.4}$$

If there are N ensembles the likelihood function is

$$L = \left(\frac{1}{P}\right)^N \exp\left\{-\frac{\Sigma OO^*}{II^*}\cdot\frac{1}{P}\right\}. \tag{4.1.5}$$

The likelihood function across a discrete frequency spectrum is

$$\prod_{\omega}\left(\frac{1}{P_{(\omega)}}\right)^n \exp\left\{-\left\{\frac{\Sigma OO^*}{II^*}\cdot\frac{1}{P_{(\omega)}}\right\}\right\} \tag{4.1.6}$$

Maximising (4.1.6) is equivalent to minimising

$$\sum_{\omega}\left(\log P_{(\omega)} + \frac{OO^*}{IIP_{(\omega)}}\right). \tag{4.1.7}$$

A unique solution for the transfer function is not possible even if $P_{(\omega)}$ is known without error. This is because if $T_{(s)} = N_{(s)}/D_{(s)}$ where N and D are polynomials in s (the Laplace variable) then

$$P(s)\bigg|_{s=i\omega} = \frac{N(s)\,N(-s)}{D(s)\,D(-s)}\bigg|_{s=i\omega} \tag{4.1.8}$$

$D(s)$ is uniquely determined since the system is stable. $N(s)$ is not uniquely determined since any pair of conjugate zeros can be replaced by their "unstable" images and still have the same power spectrum. If there are $4N$ complex zeros in $P(s)$ this means that there are $2\binom{2N}{N}$ possible solutions for $T(s)$. However the damping and frequencies are uniquely determined from $D(s)$. As an example of this non-uniqueness consider the two time histories:

(i) $-e^{-3t}+2e^{-4t}$;

(ii) $5e^{-3t}-6e^{-4t}$.

Both these time histories have the power spectrum density $\frac{(4+\omega^2)}{(9+\omega^2)(16+\omega^2)}$. Multiplication of the above time histories by -1 will generate another two time histories still having the power spectrum density $\frac{4+\omega^2}{(9+\omega^2)(16+\omega^2)}$. It should be noticed that the exponents remain the same in all four time histories. The minimum phase system is $-e^{-3t}+2e^{-4t}$.

4.2 Numerical Solution of the Likelihood Equation

After appropriate scaling the function to be minimised is

$$\left\{\sum_{\omega}\left[\log P_{(\omega)} + \frac{\overline{OO^*}}{II^*}\frac{1}{P^{(\omega)}}\right]\right\}\Big/ M = F(\alpha_1,\alpha_2,\alpha_p) \tag{4.2.1}$$

where M is the number of frequency points. For ease in notation define $S_{(\omega)} = \frac{\overline{OO^*}}{II^*}$ where $\overline{OO^*}$ is the measured averaged output power and II^* is the known average population input power. The equations to be satisfied are

$$\frac{\partial F}{\partial \alpha_i} = 0 \qquad i = 1, \ldots, p$$

If α_i are initial estimates then $\alpha_i+\Delta\alpha_i$ are improved estimates where

$$\frac{\partial F}{\partial \alpha_i} + \Sigma\, \frac{\partial^2 F}{\partial\alpha_i\partial\alpha_j}\,\Delta\alpha_j = 0 \tag{4.2.3}$$

Now

$$\frac{\partial F}{\partial \alpha_i} = \Sigma\,\frac{1}{M}\left\{\frac{1}{p}\frac{\partial P}{\partial \alpha_i} - \frac{S}{p^2}\frac{\partial P}{\partial \alpha_i}\right\}. \tag{4.2.4}$$

and

$$\frac{\partial^2 F}{\partial \alpha_i \partial \alpha_j} \doteq \Sigma \frac{1}{M}\left\{\frac{1}{p^2}\frac{\partial P}{\partial \alpha_i}\frac{\partial P}{\partial \alpha_j}\right\} \tag{4.2.5}$$

The Hessian matrix is again approximated by a globally positive definite matrix. The matrix is replaced by its expected value, and the approximation is asymptotically exact.

4.3 Linear System Excited by an Unknown Random Pressure Field

If a linear system is excited by an unknown random pressure field, not necessarily white in space and time, then it is possible to recover the damped modes, frequencies and dampings of the system provided that the pressure distribution is assumed to be due to the one random source. In other words the pressure distribution is coherent. In theory it is only necessary to assume that any extraneous noise is uncorrelated between measuring stations. However to avoid working with an unmanageable number of parameters it is assumed that the extraneous noise has a population white power spectrum and this is of the same magnitude for all measuring stations. Let

$$O_i = T_i I + N_i \qquad i = 1, \ldots, m$$

where O_i is the measured output at position i due to a random pressure distribution I and noise N_i. T_i is an overall transfer function for the output at position i and

$$E(N_i N_j^*) = \sigma^2 \delta_{ij} \tag{4.3.1}$$

$$E(II^*) = 1 \qquad E(IN_i^*) = 0 \tag{4.3.2}$$

If $E(II^*) \neq 1$ then this is absorbed in T_i. Hence

$$E(O_i O_j^*) = T_i T_j^* + \sigma^2 \delta_{ij}. \tag{4.3.3}$$

To uncorrelate the data it is necessary to find the eigenvectors of the matrix (4.3.3). The matrix (4.3.3) has a degeneracy of order $m-1$ and as a result the eigenvectors may be defined in a number of ways. This is now done.

Define

$$Z_1 = \Sigma T_i^* O_i \tag{4.3.4}$$

$$Z_m = T_1 O_m - T_m O_1 \qquad M \geqslant 2 \tag{4.3.5}$$

then

$$E(Z_1 Z_1^*) = (\Sigma T_i T_i^*)(\Sigma T_i T_i^* + \sigma^2) \tag{4.3.6}$$

$$E(Z_m Z_m^*) = (T_1 T_1^* + T_n T_n^*)\sigma^2 \tag{4.3.7}$$

$$E(Z_i Z_j^*) = 0 \qquad i \neq j.$$

The Z random variables are uncorrelated. The maximum likelihood method in this situation leads to the *minimisation of*

$$\sum_{\omega}\Bigg\{\ln(\Sigma T_i T_i^*) + \ln(\Sigma T_i T_i^* + \sigma^2) + (n-1)\ln\sigma^2 + \sum_{n=2}^{m}\left[\ln(T_1 T_1^* + T_n T_n^*) + \frac{Z_n Z_n^*}{(T_1 T_1^* + T_n T_n^*)}\sigma^2\right] + \frac{Z_1 Z_1^*}{(\Sigma T_i T_i^*)(\Sigma T_i T_i^* + \sigma^2)}\Bigg\}.$$

It should be noted that the ambiguity in the transfer functions is resolved by using the cross-power spectra (see Appendix A).

5. CONCLUSION

The maximum likelihood method has been applied to the estimation of linear dynamic system parameters for both deterministic and random excitation.

Globally stable numerical methods have been presented for the estimation of these parameters together with an estimate of the corresponding asymptotic variance. These estimatse have the desirable property of being both asymptotically unbiased and asymptotically efficient.

APPENDIX A

Ambiguity in Transfer Function Resolved Using Cross-power Spectra

Let $T_1(s)$ and $T_2(s)$ be two transfer functions from a multiple output and multiple input model. Let I be the random input to each (this is assumed to be white). Hence,

$$O_1 = T_1 I \qquad O_2 = T_2 I \tag{A1}$$

$$E(O_1 O_1^*) = T_1 T_1^* E(II^*) = T_1 T_1^* \tag{A2}$$

$$E(O_2 O_2^*) = T_2 T_2^* E(II^*) = T_2 T_2^*. \tag{A3}$$

T_1 and T_2 will have common poles and these may be identified from either the output power spectrum of channel 1 or the output power spectrum of channel 2 using the knowledge that the system is stable. Given the output power spectrum the zeros of TT^* may be determined but there is a question as to how to assign these zeros to T and T^* respectively.

Let

$$T_1 = N_1/D \text{ and } T_2 = N_2/D$$

where N_1, N_2 and D are polynomials in S. It follows that

$$E(O_2 O_1^*) = N_2 N_1^*/D^2. \tag{A4}$$

Hence

$$E(O_2 O_1^*)/E(O_2 O_2^*) = N_2^*/N_1^*. \tag{A5}$$

It follows that provided T_1 and T_2 have no zeros in common then T_1 and T_2 may be determined from power spectra and cross-power spectra. In practice the division (A5) is not performed as the argument may proceed from common factors of $E(O_2 O_1^*)$ and $E(O_2 O_2^*)$. Overall ambiguity in sign remains since multiplying each transfer function by -1 does not change the power spectra.

APPENDIX B

Redundant Information Contained in a Transfer Function Matrix

Consider a linear time invariant vibrational system obeying Maxwell's law of reciprocity with no external energy source. Let $T_{ij}(s)$ $(i = 1, N; J = 1, \ldots, N)$ be a transfer function matrix then the following statements are true.

(i) Each element has the same poles which lie on the LHS of the complex plane.

(ii) All the zeros of T_{ii} lie on the LHS of the complex plane, that is they are minimum phase transfer functions.

(iii) Any principal sub-determinant forms a minimum phase transfer function.

(iv) Knowledge of the zeros along the main diagonal determines all the zeros of all the off-diagonal elements.

Comments will now be made on each of these statements.

(i) The location of poles gives information on the damping and frequency of modes.
It is possible for zero-pole cancellation, if this occurs then for our purposes the zero and pole are both replaced.

(ii) The zeros of T_{ii} correspond to the dampings and frequencies of the original elastic system rigidly constrained at position i. Furthermore these frequencies must be greater than or equal to the frequencies of the unconstrained system, for lightly damped systems.

(iii) The zeros of any principal sub-determinant correspond to the original elastic system constrained at every position whose element is left along the main diagonal. As the determinant increases in size the transfer function tends to an all-pole filter.

(iv) As an example consider the 2×2 determinant

$$\begin{vmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{vmatrix} \quad \text{let } T_{ij} = N_{ij}/D \ (T_{12} = T_{21})$$

where N and D are polynomials in s. The determinant is

$$(N_{11}N_{22} - N_{12}^2)/D^2(s) = \frac{E(s)}{D(s)}$$

hence

$$N_{12}^2(s) = N_{11}N_{22} - E(s)D(s)$$

and N_{12}^2 may be uniquely determined, provided N_{11}, N_{22} and D are known. $E(s)$ is determined from the fact that $N_{11}N_{22} - ED$ has repeated roots and

$$N_{12}^2(S_i) = N_{11}(S_i)N_{22}(S_i)$$

where S_i is any zero of $D(s)$. The zeros of any off-diagonal term (T_{ij}) are calculated by considering the binary system

$$\begin{vmatrix} T_{ii} & T_{ij} \\ T_{ij} & T_{jj} \end{vmatrix}$$

The total transfer function matrix cannot, however, be reconstructed due to ambiguity in signs. In the example given N_{12}^2 may be uniquely found, N_{12}, however, has an ambiguity of sign.

The number of possibilities considering only binary systems is $2^{\binom{n}{2}}$. Taking into account higher order determinants reduces this number to 2^{n-1}. As an example of these "phase" possibilities consider a matrix which is 3×3 with an acceptable phase distribution

$$\begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}$$

then the other possibilities are

$$\begin{bmatrix} + & - & - \\ - & + & + \\ - & + & + \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad \begin{bmatrix} + & + & - \\ + & + & - \\ - & - & + \end{bmatrix}$$

This pattern of signs leaves all principal sub-determinants invariant.

REFERENCES

1. Kendall and Stuart, *The Advanced Theory of Statistics*, Vol. 2. Charles Griffin & Co., 1963.

2. Brillinger, D. R., *Time Series Data Analysis and Theory*. Holt Rinehart and Winston, 1975.

3. Anderson, T. W., *The Statistical Analysis of Time Series*. John Wiley & Sons, 1971.

4. Astrom, K. J., and Dykhoff, P. "System Identification." *A Survey Automatica*, Vol. 7, pp. 123–62, 1971.

5. Barlett, M. S. "Periodgram Analysis and Continuous Spectra." *Biometrika* 37, 1–16, 1950.

6. Kay, S. M., and Marple, S. L. "Spectrum Analysis—A Modern Perspective." *Proc. IEEE*, Vol. 69, No. 11, November 1981.

7. Anderson, B., and Moore, J. *Linear Optimal Control*. Prentice-Hall, 1971.

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The maximum likelihood method is used to estimate dynamic system parameters when a linear system is excited by random or deterministic inputs.

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